

# THEORY OF HADRONIC ATOMS<sup>a</sup>

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A review of the modern theory of hadronic atoms based on the non-relativistic effective Lagrangian approach and ChPT, is given. As an application of the general framework, we consider the decay of the  $\pi^+\pi^-$  atom into two neutral pions, and the energy-level shift of the  $\pi^-p$  atom. The analysis of the decay width of the  $\pi^+\pi^-$  atom has been carried out at the first non-leading order in isospin breaking, and at  $O(e^2p^2)$  in ChPT, that results in an accurate theoretical prediction for this quantity. The study of the  $\pi^-p$  atom energy-level shift within the same approach clearly demonstrates the necessity to critically reassess the values of the  $\pi N$  scattering lengths, extracted from the energy-level shift measurement by means of the potential model-based theoretical analysis. The construction of short-range hadronic potentials from field theory is also discussed.

## 1 Introduction

Recent years have seen a growing interest in the study of hadronic bound states - so called hadronic atoms - that provides an extremely valuable piece of information on the behavior of QCD at a very low energy. At CERN, the DIRAC collaboration<sup>1</sup> aims to measure the  $\pi^+\pi^-$  atom lifetime to 10% accuracy. This would allow one to determine the difference  $a_0 - a_2$  of  $\pi\pi$  scattering lengths with 5% precision. This measurement provides a crucial test for the large/small condensate scenario in QCD: should it turn out that the quantity  $a_0 - a_2$  is different from the value predicted in standard ChPT<sup>2</sup>, one has to conclude<sup>3</sup> that spontaneous chiral symmetry breaking in QCD proceeds differently from the widely accepted picture. In the experiment performed at PSI<sup>4,5</sup>, one has measured the strong energy-level shift and the total decay width of the  $1s$  state of pionic hydrogen, as well as the  $1s$  shift of pionic deuterium. These measurements yield isospin symmetric  $\pi N$  scattering lengths to an accuracy which is unique for hadron physics. A new experiment on pionic hydrogen at PSI has recently been approved. It will allow one to measure the decay  $A_{\pi^-p} \rightarrow \pi^0 n$  to much higher accuracy and thus enable one, in principle, to determine the  $\pi N$  scattering lengths from data on pionic hydrogen alone. This might vastly reduce the model-dependent uncertainties that come from the analysis of the three-body problem in  $A_{\pi^-d}$ . Finally, the DEAR

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collaboration<sup>6</sup> at the DAΦNE facility plans to measure the energy level shift and lifetime of the  $1s$  state in  $K^-p$  and  $K^-d$  atoms - with considerably higher precision than in the previous experiment carried out at KEK<sup>7</sup> for  $K^-p$  atoms. It is expected<sup>6</sup> that this will result in a precise determination of the  $I = 0, 1$   $S$ -wave scattering lengths. It will be a challenge for theorists to extract from this new information on the  $\bar{K}N$  amplitude at threshold a more precise value of e.g. the isoscalar kaon-sigma term and of the strangeness content of the nucleon.

In order to carry out the precision experimental tests of QCD mentioned above, on the theoretical side one faces the problem of finding the suitable field-theoretical framework for the description of the measured characteristics of hadronic atoms - energy levels and decay probabilities. In this work we shall report on the recent progress in this direction. The main message which will be delivered, is that a rigorous theory of this sort of bound states based on the merger of ChPT and the non-relativistic effective Lagrangian technique that was proposed originally by Caswell and Lepage<sup>8</sup> to study QED bound states in general, can be indeed constructed. This theory allows for a systematic expansion of the bound-state observables both in quark mass  $\hat{m} = \frac{1}{2}(m_u + m_d)$ , and the isospin-breaking parameters: fine structure constant  $\alpha$  and the quark mass difference  $m_d - m_u$  (for simplicity, we consider an  $SU(2)$  case here).

Hadronic atoms appear to be loosely bound systems of hadrons that are formed mainly by the static Coulomb force. The Bohr radius of this sort of bound states is of order of a few hundreds of Fm, and the average 3-momenta of constituents lie in MeV range. For the above reason, it is evident that the non-relativistic framework provides the most natural and economical tool for handling such bound states. Relativistic corrections are taken into account perturbatively, up to any given order in the expansion in inverse powers of masses. Below, we briefly outline the main ingredients of the non-relativistic effective Lagrangian approach to bound states, without going into details.

- The non-relativistic Lagrangian describing interactions between hadrons and photons, is built from the non-relativistic hadron fields and the photon field. This Lagrangian consists of an infinite tower of all possible operators, with an increasing mass dimension - all operators allowed by discrete symmetries and the gauge invariance should be included, with an *a priori* unknown couplings. In actual calculations, only a few low-dimensional ones matter - higher-dimensional operators contribute to higher powers in  $\alpha$  in the bound-state observables. This feature of the effective theory goes under the name of “power counting”. Further, the Lagrangian does not include, by definition, the operators that change the number of heavy particles (hadrons).

- Loop corrections to the scattering amplitudes in the non-relativistic the-

ory are calculated in a standard manner, by using the Feynman diagrammatic technique. There is, however, one important modification. It is well known that in the non-relativistic theory in the presence of light particles (photons) the Feynman integrals should be properly regularized in order to avoid the contribution from the loop momenta at the hard scale - otherwise, loop corrections to the Green functions would lead to the breakdown of counting rules in the non-relativistic theory. A suitable regularization procedure built on the top of the Feynman rules in the non-relativistic theory is provided by so-called “threshold expansion”<sup>9</sup>, that enables one to disentangle the contributions coming from different regions of loop momenta, by expanding the integrands - in the dimensional regularization - in all possible small kinematical variables. Next, one has to systematically remove hard-momentum contribution from the integrals, which at low energies is given by a polynomial in external momenta, and can be absorbed into the renormalization of the couplings in the non-relativistic Lagrangian. For a more detailed discussion of the problem, see<sup>10</sup>.

- Couplings in the non-relativistic effective Lagrangian are determined from the matching to the relativistic theory. These couplings are not necessarily real, since the physical decay processes where the number of the hadrons is not conserved, contribute to the imaginary parts of these couplings. The matching condition determines the couplings in terms of threshold parameters of the physical hadronic scattering amplitudes. It is crucial to stress that the matching condition does not imply the chiral expansion of the amplitudes - formally, it is valid in all orders in the chiral expansion.

- After setting the parameters in the non-relativistic Lagrangian, we turn to the bound states in the theory. The Feshbach formalism<sup>11</sup> that allows one to separate the bound-state pole in the scattering amplitudes, turns out to be very convenient for this purpose. The real and imaginary parts of the pole position on the second Riemann sheet of the complex energy plane coincide, by definition, with the energy and the decay width of the (metastable) bound state. The perturbative framework for determining the pole position coincides with the conventional Rayleigh-Schrödinger perturbation theory. At the end, the characteristics of the bound state are determined through the couplings in the non-relativistic amplitudes. With the use of the matching condition, these characteristics can be further expressed via the scattering amplitudes in the relativistic theory.

- Given the chiral expansion for the scattering amplitude, it is possible to obtain the chiral expansion for the bound-state observables which can be reorganized in the expansion in  $\hat{m}$  and isospin-breaking parameters. The isospin symmetry world is, by convention, defined as the one where the masses of pions

an nucleons coincide with the charged ones, and the values of all low-energy constants remain the same. In this vein, one may achieve an unambiguous separation of the isospin-breaking effects and extract the isospin-symmetric hadronic scattering lengths directly from the hadronic atom measurements.

After having described the general framework, we consider to the particular systems.

## 2 Decay of the $\pi^+\pi^-$ atom into $\pi^0\pi^0$

Recently, using the non-relativistic effective Lagrangian framework, a general expression for the decay width  $\Gamma_{A_{2\pi} \rightarrow \pi^0\pi^0}$  of the  $1s$  state of the  $\pi^+\pi^-$  atom was obtained at next-to-leading order in isospin-breaking<sup>12</sup>. Numerical analysis of this quantity was carried out at order  $O(e^2p^2)$  in ChPT<sup>13</sup>. These investigations have confirmed and generalized the results of earlier studies<sup>14,15</sup>. The expression for the decay width at the first non-leading order in isospin breaking has the form

$$\Gamma_{A_{2\pi} \rightarrow \pi^0\pi^0} = \frac{2}{9} \alpha^3 p^\star \mathcal{A}_{\pi\pi}^2 (1 + K_{\pi\pi}), \quad (1)$$

$$K_{\pi\pi} = \frac{\Delta M_\pi^2}{9M_{\pi^+}^2} (a_0 + 2a_2)^2 - \frac{2\alpha}{3} (\ln \alpha - 1)(2a_0 + a_2) + o(\alpha, (m_d - m_u)^2).$$

Here  $p^\star = (M_{\pi^+}^2 - M_{\pi^0}^2 - \frac{1}{4}M_{\pi^+}^2\alpha^2)^{1/2}$ , and  $a_I$ , ( $I = 0, 2$ ) denote the (dimensionless) strong  $S$ -wave  $\pi\pi$  scattering lengths in the channel with total isospin  $I$ . The quantity  $\mathcal{A}_{\pi\pi}$  is obtained as follows<sup>12</sup>. One calculates the relativistic amplitude for the process  $\pi^+\pi^- \rightarrow \pi^0\pi^0$  at  $O(\alpha, (m_d - m_u)^2)$  in the normalization chosen so that at  $O(1)$  the amplitude at threshold coincides with the difference  $a_0 - a_2$ . Due to the presence of virtual photons, the amplitude is multiplied by an overall Coulomb phase  $\theta_c$  that is removed. The real part of the remainder contains terms that diverge like  $|\mathbf{p}|^{-1}$  and  $\ln 2|\mathbf{p}|/M_{\pi^+}$  at  $|\mathbf{p}| \rightarrow 0$  ( $\mathbf{p}$  denotes the relative 3-momentum of charged pion pairs). The quantity  $\mathcal{A}_{\pi\pi}$  is obtained by subtracting these divergent pieces, and by then evaluating the remainder at  $\mathbf{p} = 0$ .

$$\text{Re}(e^{-i\theta_c} t_{\pi\pi}) \rightarrow \frac{b_1}{|\mathbf{p}|} + b_2 \ln \frac{2|\mathbf{p}|}{M_{\pi^+}} + \frac{8\pi}{3M_{\pi^+}^2} \mathcal{A}_{\pi\pi} + \dots \quad (2)$$

As it is seen explicitly from Eq. (1), one can directly extract the value of  $\mathcal{A}_{\pi\pi}$  from the measurement of the decay width, because the correction  $K_{\pi\pi}$  is very small and the error introduced by it is negligible. In order to extract strong scattering lengths from data, one may invoke ChPT and to relate the quantities

$\mathcal{A}_{\pi\pi}$  and  $a_0 - a_2$  order by order in the chiral expansion<sup>13</sup>. This requires the evaluation of isospin-breaking corrections to the scattering amplitude. At order  $O(e^2 p^2)$  in chiral expansion we obtain

$$A_{\pi\pi} = a_0 - a_2 + \epsilon_{\pi\pi}, \quad \epsilon_{\pi\pi} = (0.58 \pm 0.16) \cdot 10^{-2}, \quad K_{\pi\pi} = 1.15 \cdot 10^{-2}. \quad (3)$$

The lifetime of the  $\pi^+\pi^-$  atom is predicted to be

$$\tau_{2\pi^0} = (2.91 \pm 0.09) \cdot 10^{-15} \text{ s}, \quad (4)$$

and the correction to the leading-order in isospin-breaking  $\Gamma_{2\pi^0} = \Gamma_{2\pi^0}^{LO}(1 + \delta_\Gamma)$  where  $\Gamma_{2\pi^0}^{LO} = \frac{2}{9} \alpha^3 p^* (a_0 - a_2)^2$ , equals to  $\delta_\Gamma = 0.056 \pm 0.012$ . Note that above we have used the values of  $\pi\pi$  scattering lengths  $a_0 = 0.220$ ,  $a_2 = -0.0444$ ,  $\Delta(a_0 - a_2) = 0.004$ , obtained on the basis of two loop calculations in ChPT and dispersion relations analysis<sup>16</sup>. For the various low-energy constants entering the expression for  $\epsilon_{\pi\pi}$ , the same values as in Ref.<sup>13</sup> were used. These values of the low-energy constants will not necessarily coincide with the ones that will emerge from the analysis of scattering lengths. In addition, the errors in the scattering lengths and the correction term at this stage are treated as uncorrelated. Only once the new values for the low-energy constants  $\bar{l}_i$  from the dispersion analysis are available including error bars, one may refine the above preliminary analysis for the decay width. We also emphasize, that the corrections to the decay width at  $O(e^2 p^2)$  are already very small, justifying the negligence of the higher-order terms<sup>13</sup>.

### 3 Energy level shift of the $\pi^-p$ atom

The treatment of the  $\pi^-p$  atom problem<sup>17</sup> proceeds along the lines very similar to those for  $\pi^+\pi^-$  case. Our investigations are aimed at the derivation of the general expression for the  $\pi^-p$  atom energy-level shift in the  $1s$  state. The total shift is given by a sum of the electromagnetic and strong pieces. Our calculations for the electromagnetic shift<sup>17</sup> within a high accuracy yield the same result as given in Ref.<sup>4</sup>. The final result for the strong shift  $\epsilon_{1s}$  in the first non-leading order in isospin breaking is given in a form similar to Eq. (1)

$$\epsilon_{1s} = -2\alpha^3 \mu_c^2 \mathcal{A}_{\pi N} (1 + K_{\pi N}) \quad (5)$$

$$K_{\pi N} = -2\alpha \mu_c (\ln \alpha - 1) \mathcal{A}_{\pi N} + o(\alpha, m_d - m_u),$$

where  $\mu_c$  denotes the reduced mass of the  $\pi^-p$  pair, and the quantity  $\mathcal{A}_{\pi N}$  is defined analogously to  $\mathcal{A}_{\pi\pi}$ . To calculate this quantity, one has to evaluate the  $\pi^-p \rightarrow \pi^-p$  relativistic scattering amplitude at  $O(\alpha, m_d - m_u)$ , drop all

diagrams that are made disconnected by cutting one photon line, and discard the spin-flip piece. The remainder is denoted by  $t_{\pi N}$ . The regular part of  $t_{\pi N}$  at threshold defines the quantity  $\mathcal{A}_{\pi N}$  in analogy to Eq. (2)

$$\text{Re}(e^{-2i\theta_c} t_{\pi N}) \rightarrow \frac{B_1}{|\mathbf{p}|} + B_2 \ln \frac{|\mathbf{p}|}{\mu_c} - \frac{2\pi}{\mu_c} \mathcal{A}_{\pi N} + \dots \quad (6)$$

In order to extract the value of the  $S$ -wave  $\pi N$  scattering lengths  $a_{0+}^+$ ,  $a_{0+}^-$  from the  $\pi^- p$  energy shift measurement, one may again resort to ChPT, to calculate the isospin-breaking corrections to the  $\pi N$  scattering amplitude at threshold. The normalization of the relativistic amplitude is chosen so that at threshold  $\mathcal{A}_{\pi N} = a_{0+}^+ + a_{0+}^- + O(\alpha, m_d - m_u)$ . We have carried out these calculations at chiral order  $O(p^2)$ , where only the tree diagrams contribute, the result looks as follows

$$\mathcal{A}_{\pi N} = a_{0+}^+ + a_{0+}^- + \epsilon_{\pi N}, \quad \epsilon_{\pi N} = \frac{m_p(8c_1\Delta M_\pi^2 - 4e^2f_1 - e^2f_2)}{8\pi(m_p + M_{\pi^+})F^2}, \quad (7)$$

where  $c_i$  ( $f_i$ ) are the strong (electromagnetic) low-energy constants from the  $O(p^2)$  Lagrangian of ChPT<sup>18</sup>. In order to perform the numerical analysis, one has to specify the values of these low-energy constants. The "strong" constant  $c_1$  can be determined from the fit of the elastic  $\pi N$  scattering amplitude at threshold to KA86 data<sup>19</sup>:  $c_1 = -0.925 \text{ GeV}^{-1}$ . The value of the constant  $f_2$  can be extracted from the proton-neutron electromagnetic mass difference<sup>20</sup>:  $e^2f_2 = (-0.76 \pm 0.3) \text{ MeV}$ . The determination of the constant  $f_1$  from data is however, problematic. For this reason, in our analysis we have used order-of-magnitude estimate for this constant:  $-|f_2| \leq f_1 \leq |f_2|$ . With these values of the low-energy constants, we obtain the isospin-breaking correction to the leading-order energy-level shift defined as  $\epsilon_{1s} = \epsilon_{1s}^{LO}(1 + \delta\epsilon)$ , where  $\epsilon_{1s}^{LO} = -2\alpha^3\mu_c^3(a_{0+}^+ + a_{0+}^-)$ , to be  $\delta\epsilon = (-4.8 \pm 2.0) \cdot 10^{-2}$ . The large uncertainty is caused mainly by the poor knowledge of the parameter  $f_1$ . In demonstration of the above discussion, in Fig. 1 we confront the results of the above analysis with those of the potential model<sup>5</sup>:  $\delta\epsilon = (-2.1 \pm 0.5) \cdot 10^{-2}$ , using the same experimental input. As it is readily seen from Fig. 1, the systematic error in the potential approach is grossly underestimated, that already indicates at the necessity to critically reassess the values of the  $S$ -wave  $\pi N$  scattering lengths quoted in Ref. <sup>5</sup>. In addition, it remains to be seen, how our results will be altered by the loop corrections in ChPT. A precise determination of the low-energy constant  $f_1$ , using either sum rules or invoking various models, is also desirable.

As it was mentioned in the Introduction, a new experiment at PSI is approved, that is aimed at a precise determination of the  $\pi^- p$  atom decay width.

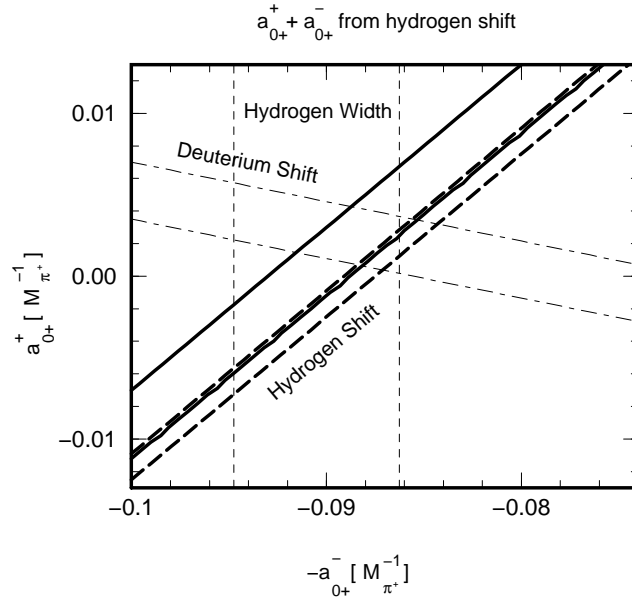


Figure 1: *Determination of  $\pi N$  scattering lengths from the pionic hydrogen and pionic deuterium measurements. Solid line corresponds to the energy-level shift calculations at  $O(p^2)$  in ChPT, and the dashed lines - to the potential model results*

At the first non-leading order in isospin breaking, the width is determined by the charge-exchange amplitude  $\pi^- p \rightarrow \pi^0 n$  where the low-energy constant  $f_1$  does not appear. Consequently, the calculations for this quantity can be carried out with a less theoretical uncertainty.

#### 4 Construction of the potentials from field theory

The study of the properties of the hadronic atoms within the potential approach has a decades-long history. As a general rule, the predictions made within the potential model badly deviate from the ones obtained within the field-theoretical approach based on ChPT. For example, the potential model predicts the isospin-breaking correction  $\delta_{\Gamma}$  to the  $\pi^+ \pi^-$  atom decay width with an opposite sign and with the same order of magnitude<sup>21</sup>. The predictions for the  $\pi^- p$  atom energy-level shift<sup>4</sup> have been already considered in the previous section. The reason for this discrepancy is now well understood. Namely, the potential model does not take into account a full content of isospin-symmetry

breaking in QCD. In particular, the effect of the direct quark-photon interactions (encoded in the “electromagnetic” low-energy constants of ChPT), as well as the effect stemming from the dependence of scattering amplitudes on the quark masses, are not included. As these effects contribute a bulk of the total isospin-breaking correction term in the field-theoretical approach, it does not come to our surprise that the predictions of both approaches substantially differ.

Since the isospin-breaking effects discussed above lead, in the language of the potential scattering theory, to the modification of the short-range part of the hadronic potential, it is evident that in order to bring the potential model in conformity with the field-theoretical treatment, one has to assume that short-range hadronic potentials contain an isospin-breaking piece. It is natural to seek a derivation of the potentials that are used in the potential model, on the basis of ChPT. In a slightly more restricted context, one may ask, how the isospin-breaking part of the short-range “strong” potential is obtained from ChPT, when the isospin-symmetric part is already known to fit well ChPT predictions (we recall that the isospin-breaking part is assumed to vanish identically in existing potential models<sup>4,21</sup>).

It is widely presumed that the potential constructed from the field theory will be necessarily singular in the position space and will require some kind of regularization<sup>22</sup>. We argue that this is not necessarily the case: almost any well-behaved short-range potential, including those that were used in Refs.<sup>4,21</sup>, can be generalized to include properly the full content of isospin-breaking effects in ChPT.

The key observation that leads to the above conclusion, can be summarized in the so-called universality conjecture. This conjecture - completely in spirit of the low-energy effective Lagrangian approach to bound systems - states that the bound-state energies in the field theory, and in the potential model are the same at the first order in isospin breaking, provided the threshold amplitudes calculated in these two theories, coincide. We shall ensure the universality for the case of a simple one-channel model where the interaction Hamiltonian is given by a sum of Coulomb and short-range interactions  $\mathbf{H}_I = -\alpha r^{-1} + \mathbf{U}$ . The short-range potential  $\mathbf{U}$ , in general, contains isospin-breaking effects.

With a given interaction potential, one may evaluate the energy-level shift of the ground state of the bound system. The equation for the position of the bound-state pole in the (complex) energy plane is given by<sup>12</sup>

$$z - E_0 - \langle \Psi_0 | \tau(z) | \Psi_0 \rangle = 0, \quad (8)$$

$$\tau(z) = \mathbf{U} + \mathbf{U}(z - \mathbf{H}_0)^{-1}(1 - |\Psi_0\rangle\langle\Psi_0|)\tau(z),$$



where  $E_0$  denotes the unperturbed Coulomb energy of the ground state, and  $\Psi_0$  stands for the unperturbed wave function. The iterative solution of Eq. (8) yields usual Rayleigh-Schrödinger perturbation series. Further, it turns out useful to introduce the scattering matrix on the short-range potential only

$$\mathbf{t}(z) = \mathbf{U} + \mathbf{U} \frac{1}{z - \mathbf{H}_0} \mathbf{t}(z), \quad \mathbf{t}_0 \equiv \langle \mathbf{q} | \mathbf{t}(z) | \mathbf{p} \rangle \Big|_{\mathbf{p}=\mathbf{q}=0, z=0}. \quad (9)$$

Here  $\mathbf{t}_0$  denotes the value of the scattering amplitude at threshold (for convenience, we have shifted the threshold to  $z = 0$ ).

The perturbative solution of the equation for the bound-state energy at  $O(\alpha^4)$  gives

$$\epsilon_{1s} = \frac{\alpha^3 \mu_c^3}{\pi} \left\{ \mathbf{t}_0 - \frac{\alpha \mu_c^2}{\pi} \mathbf{t}_0^2 \left( 1 + \ln \frac{b}{2\alpha \mu_c} \right) + \frac{8\alpha \mu_c}{\pi} \mathcal{Q}[\mathbf{t}] + \frac{\alpha \mu_c^2}{\pi^3} \mathcal{R}[\mathbf{t}; b] \right\}, \quad (10)$$

where  $\mathcal{Q}$  and  $\mathcal{R}$  are certain known functionals of the scattering matrix  $\mathbf{t}$ . We do not display the explicit expressions here. The dependence on the arbitrary cutoff parameter  $b$  in the functional  $\mathcal{R}$  cancels with the logarithmic dependence in the second term, so that  $d\epsilon_{1s}/db = 0$ .

At the next step, we consider the full scattering amplitude defined through the Lippmann-Schwinger equation  $\mathbf{T}(z) = \mathbf{H}_I + \mathbf{H}_I(z - \mathbf{H}_0)^{-1} \mathbf{T}(z)$ . The threshold behavior of  $\mathbf{T}(z)$  on energy shell is given by (cf with Eq. (6))

$$\begin{aligned} \text{Re} [e^{-2i\theta_c} \langle \mathbf{p} | (\mathbf{T}(z) - \mathbf{H}_I) | \mathbf{q} \rangle] \Big|_{|\mathbf{p}|=|\mathbf{q}|, z=\mathbf{p}^2/\mu_c} &= \frac{\tilde{B}_1}{|\mathbf{p}|} + \tilde{B}_2 \ln \frac{|\mathbf{p}|}{\mu_c} \\ &- \frac{2\pi}{\mu_c} \mathcal{A} + O(|\mathbf{p}|) + o(\alpha) \end{aligned} \quad (11)$$

$$-\frac{2\pi}{\mu_c} \mathcal{A} = \mathbf{t}_0 - \frac{\alpha \mu_c^2}{\pi} \mathbf{t}_0^2 \ln \frac{b}{2\mu_c} + \frac{8\alpha \mu_c}{\pi} \mathcal{Q}[\mathbf{t}] + \frac{\alpha \mu_c^2}{\pi^3} \mathcal{R}[\mathbf{t}; b] + o(\alpha), \quad (12)$$

with the same functionals  $\mathcal{Q}$  and  $\mathcal{R}$ .

If one now expresses the energy level shift in terms of the threshold amplitude  $\mathcal{A}$ , one arrives at exactly the same expression (5) as in the field-theoretical framework - in accordance with the universality conjecture

$$\epsilon_{1s} = -2\alpha^3 \mu_c^2 \mathcal{A} (1 - 2\alpha \mu_c (\ln \alpha - 1) \mathcal{A}) + o(\alpha^4). \quad (13)$$

Based on the universality conjecture, we can provide a constructive algorithm for the derivation of the isospin-breaking part of the short-range potential  $\mathbf{U}$  from ChPT. The amplitude at threshold in the latter is generally

given by  $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + \dots$ , where  $\mathcal{A}_0, \mathcal{A}_1$  denote the isospin-conserving (breaking) parts of the amplitude, and ellipses stand for higher-order terms in isospin breaking. In order to ensure the inclusion of the full content of isospin-symmetry breaking in ChPT into the potential model, it thus suffices to match the amplitude  $\mathcal{A}$  in both theories. The problem evidently has too much degrees of freedom. The short-range potential is also given by the sum of isospin-conserving and isospin-breaking pieces  $\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_1 + \dots$ , and for our purposes the following *ansatz* is sufficient:  $\mathbf{U} = (1 + \lambda)\mathbf{U}_0$ , where the sole coupling  $\lambda$  will be determined from matching the isospin-breaking pieces of the threshold amplitude.

If one defines the scattering amplitude in the limit of no isospin breaking through  $\bar{\mathbf{t}}(z) = \mathbf{U}_0 + \mathbf{U}_0(z - \mathbf{H}_0)^{-1}\bar{\mathbf{t}}(z)$ , The matching condition may be written recursively, in terms of  $\bar{\mathbf{t}}$

$$\begin{aligned} -\frac{2\pi}{\mu_c} \mathcal{A}_0 &= \bar{\mathbf{t}}_0, \\ -\frac{2\pi}{\mu_c} \mathcal{A}_1 &= \lambda(\bar{\mathbf{t}}_0 + \mathcal{S}[\bar{\mathbf{t}}]) - \frac{\alpha\mu_c^2}{\pi} \bar{\mathbf{t}}_0^2 \ln \frac{b}{2\mu_c} + \frac{8\alpha\mu_c}{\pi} \mathcal{Q}[\bar{\mathbf{t}}] + \frac{\alpha\mu_c^2}{\pi^3} \mathcal{R}[\bar{\mathbf{t}}; b]. \end{aligned} \quad (14)$$

where  $\mathcal{S}$  again stands for a certain known functional. The matching condition (14) solves our problem completely - the bound-state energies calculated with the use of the “corrected” potential coincide, by definition, with those calculated on the basis of ChPT.

## 5 Conclusions

The approach based on non-relativistic effective Lagrangian technique and ChPT, provides one a powerful tool to systematically calculate the characteristics of loosely bound states of hadrons. With the use of this approach, the  $\pi^+\pi^-$  atom decay problem is now completely understood, both conceptually and numerically. Certain theoretical effort will be still needed to extract the precise values of  $\pi N$  scattering lengths from past and future measurements at PSI. The treatment of other systems, like kaonic atoms that will be measured by the DEAR experiment at DAΦNE, is foreseen within the same framework.

Further, within the present approach, one may establish a constructive algorithm for the derivation of the short-range hadronic potentials from ChPT. The algorithm is based on the universality conjecture that has been discussed above, for a simple one-channel case. We hope that - after the suitable generalization - the approach based on the universality might be also useful for the analysis of  $\pi N$  scattering data near threshold, in what concerns the study of

the isospin-breaking effects in the  $\pi N$  amplitude.

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